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Spectral Decomposition of Tent Maps Using Symmetry Considerations

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The spectral decompostion of the Frobenius–Perron operator of maps composed of many tents is determined from symmetry considerations. The eigenstates involve Euler as well as Bernoulli polynomials.

KEY WORDS: Frobenius–Perron operator; generalized spectral decomposition; chaotic maps; symmetry; tent maps.

1. INTRODUCTION

Experimental data from chaotic systems, whether obtained from the laboratory or computer simulations, are often presented in the form of power spectra of correlation functions. From the data various decay contributions may be identified. For a model system the decay rates may be obtained from the spectrum of the time evolution operator and the weights of the contributions may be obtained from the knowledge of the eigenstates.

The time evolution operator for probability densities in chaotic maps is known as the Frobenius–Perron operator.[1] The spectral decomposition of the Frobenius–Perron operator depends on the domain in which it is considered to act.[2] For a class of systems, if the domain is restricted to smooth functions, elements of the spectral decomposition are in a generalized functional space.[2,3] These new decompositions are quite natural from a physical point of view because the spectrum then explicitly contains the decay rates and the approach to equilibrium of correlations, even in systems with reversible trajectory dynamics, is made manifest.[2–4]

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Recently, the authors have introduced some new techniques,[5] based on symmetry considerations, enabling the construction of spectral decompositions in a much simpler way than previous construction algorithms. Here we utilize these techniques to construct the spectral decomposition for one-dimensional maps of the unit interval composed of many tents. The construction uses the knowledge of the spectral decomposition of the r-adic map, which involves Bernoulli polynomials and their duals. It will be seen that the spectral decomposition of the tent maps involves both Bernoulli polynomials and Euler polynomials along with the appropriate dual states.

2. INCOMPLETE TENT MAPS

We consider piecewise-linear maps of the unit interval composed of r branches with alternating slopes $\pm r$, each mapping onto the unit interval. By convention we take the first branch [corresponding to the interval $(0, 1/r)$] to be of positive slope and we denote the Frobenius–Perron operator corresponding to the map as U_{Tr}. In this section we consider maps composed of an odd number q of branches, such as the three-branch map shown in Fig. 1. The tent pattern of the map is then incomplete. We determine first the spectral decomposition of the three-branch map from which the general result for this class easily follows.

The Frobenius–Perron operator U_{T3} acts on a density $\rho(x)$ as

$$U_{T3}\rho(x)=\frac{1}{3}\left[\rho\left(\frac{x}{3}\right)+\rho\left(\frac{2-x}{3}\right)+\rho\left(\frac{2+x}{3}\right)\right]$$

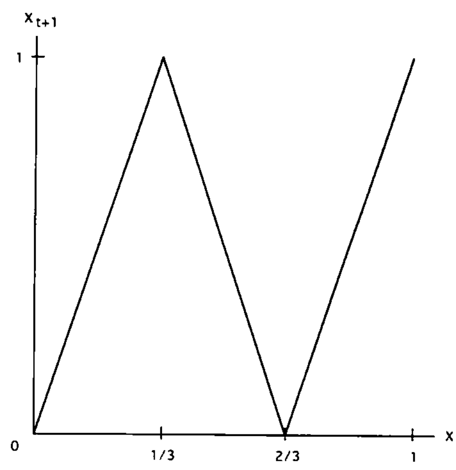

Fig. 1. The three-branch incomplete tent map.

This operator admits polynomial eigenstates, and by considering its matrix representation as acting on monomials, it is triangular so the eigenvalues are found on the diagonal. Except for the first eigenvalue of 1, corresponding to the invariant uniform density, they are found to be twofold degenerate. The eigenpolynomials of order $2n-1$ and $2n$, where $n \geqslant 1$, both have the associated eigenvalues of 3^{-2n}.

The operator U_{T3} satisfies an intertwining relation with the Frobenius–Perron operator of the 3-adic map U_3 and the two-branch tent map (commonly referred to as "the tent map") $U_T \equiv U_{T2}$ as

$$U_{T3} U_T = U_T U_3 \tag{2.1}$$

The Frobenius–Perron operator of the tent map is given by

$$U_T \rho(x) = \frac{1}{2}\left[\rho\left(\frac{x}{2}\right) + \rho\left(1 - \frac{x}{2}\right)\right]$$

The 3-adic map is a special case of the r-adic map (r being a positive integer) which is composed of r linear branches all of slope $+r$. The associated Frobenius–Perron operator U_r is given by

$$U_r \rho(x) = \frac{1}{r} \sum_{i=0}^{r-1} \rho\left(\frac{x+i}{r}\right)$$

The intertwining relation (2.1) is useful because the spectral decomposition of U_r is known.[2–4] The eigenpolynomials (right eigenstates) are the Bernoulli polynomials $B_n(x)$ defined by the generating function

$$\frac{p e^{xp}}{e^p - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} p^n$$

The left eigenstates of U_r (right eigenstates of $U_r^\dagger$) are the generalized functions

$$\tilde{B}_n(x) = \frac{(-1)^{n-1}}{n!} [\delta^{(n-1)}(x-1) - \delta^{(n-1)}(x)]$$

These are the duals of the Bernoulli polynomials as

$$\int_0^1 dx\, \tilde{B}_n^*(x)\, B_m(x) = \delta_{nm}$$

where integration with respect to Lebesgue measure over the unit interval defines the inner product. The associated eigenvalue of both $B_n(x)$ and

$\tilde{B}_n(x)$ is r^{-n}. The eigenstates of U_r are also eigenstates of the reflection operator R defined by $Rf(x) \equiv f(1-x)$. This can be seen from the fact that R and U_r commute.

Operating on both sides of (2.1) by an even-order Bernoulli polynomial $B_{2n}(x)$, and using that it is an eigenstate of U_3 with eigenvalue 3^{-2n}, we then obtain

$$U_{T3} U_T B_{2n}(x) = 3^{-2n} U_T B_{2n}(x)$$

Thus, $U_T B_{2n}(x) = B_{2n}(x/2)$ is an eigenpolynomial of U_{T3} with eigenvalue 3^{-2n}. Since the eigenpolynomial of order $2n-1$ has the same eigenvalue we may add any multiple of this lower-order polynomial to $B_{2n}(x/2)$ to obtain different eigenstates of order $2n$.

The intertwining relation (2.1) is not directly useful to obtain the odd-order eigenstates because for the odd-order Bernoulli polynomials we have $U_T B_{2n+1}(x) = 0$. We may obtain the odd-order eigenstates using that U_{T3} commutes with R. Realizing this tells us that $RB_{2n}(x/2)$ is also an eigenstate of U_{T3}, with eigenvalue 3^{-2n}. Hence any linear combination of $B_{2n}(x/2)$ and $RB_{2n}(x/2)$ is an eigenstate as well. The simultaneous eigenstates of two commuting operators form a complete set if the operators considered separately admit complete sets of eigenstates.[6] Hence, we may form a complete set of right eigenstates of U_{T3} by making them a complete set of eigenstates of R also. Such states may be constructed using the projection operators $P^{\pm} \equiv (1 \pm R)/2$. Thus, out of all the possible linear combinations mentioned above, we choose

$$P^+ B_{2n}\left(\frac{x}{2}\right) = 2^{-2n} B_{2n}(x)$$

and

$$P^- B_{2n}\left(\frac{x}{2}\right) = -n 2^{-2n} E_{2n-1}(x)$$

where $E_{2n-1}(x)$ is the Euler polynomial of order $2n-1$. (For the even-order states n starts at 0 and for the odd-order states at 1.) The Euler polynomials are defined by the generating function[7]

$$\frac{2e^{xp}}{e^p+1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} p^n$$

For convenience we choose the coefficient of the highest power of x in the eigenpolynomial to be 1. The eigenpolynomials of the incomplete tent map U_{T3} are thus taken as $I_{2n}(x) \equiv B_{2n}(x)$ and $I_{2n-1}(x) \equiv E_{2n-1}(x)$.

We have already mentioned the states, $\tilde{B}_n(x)$ which form an orthonormal set with the Bernoulli polynomials. The dual states of the full set of Euler polynomials may be determined from integration over the unit interval with the generating function. They are

$$\tilde{E}_n(x) = \frac{(-1)^n}{2(n!)} [\delta^{(n)}(x-1) + \delta^{(n)}(x)]$$

The even-order states $\tilde{B}_{2n}(x)$ are all orthogonal to the odd-order Euler polynomials and the odd-order states $\tilde{E}_{2n-1}(x)$ are all orthogonal to the even-order Bernoulli polynomials. Thus, the complete set of orthonormal duals is given by $\tilde{I}_{2n}(x) \equiv \tilde{B}_{2n}(x)$ and $\tilde{I}_{2n-1}(x) \equiv \tilde{E}_{2n-1}(x)$.

We may follow in parallel all of the above arguments to obtain the spectral decomposition of any incomplete tent map. For the incomplete tent map with q branches the eigenvalues are twofold degenerate, with the eigenvalue q^{-2n} corresponding to both the order-$2n$ and order-$(2n-1)$ eigenpolynomials. An intertwining relation like (2.1) but with the q-adic map is satisfied for tent maps with q branches, and the Frobenius–Perron operator commutes with R, so we obtain the same eigenstates as given above for the three-branch case.

As mentioned, the eigenstates of $U_{\mathrm{T}q}$ obtained above are eigenstates of R as well. Thus, they are also eigenstates of $RU_{\mathrm{T}q} = U_{\mathrm{T}q}R$, which is the Frobenius–Perron operator corresponding to an incomplete tent map with the first branch having negative slope. The odd-order eigenstates $I_{2n-1}(x)$ here have the corresponding eigenvalues $-q^{-2n}$, since $RI_{2n-1}(x) = -I_{2n-1}(x)$. The even-order eigenstates $I_{2n}(x)$ have the same eigenvalues as given above, since $RI_{2n}(x) = I_{2n}(x)$ Hence, in these maps there are no repeated eigenvalues.

We note that the above results for the eigenpolynomials of an incomplete tent map may also be obtained by considering symmetry transformations of the multiplication theorems[7] for both the Bernoulli and Euler polynomials.

3. COMPLETE TENT MAPS

We now consider maps composed of an even number p of linear branches so that the tent pattern of the map is complete. Figure 2 shows a six-branch map from this class. The eigenvalues of the polynomial eigenstates may be obtained as before by considering the Frobenius–Perron operator acting on monomials. Here the even-order eigenstates have the associated eigenvalues of p^{-2n} and the odd-order eigenstates all have eigenvalue 0.

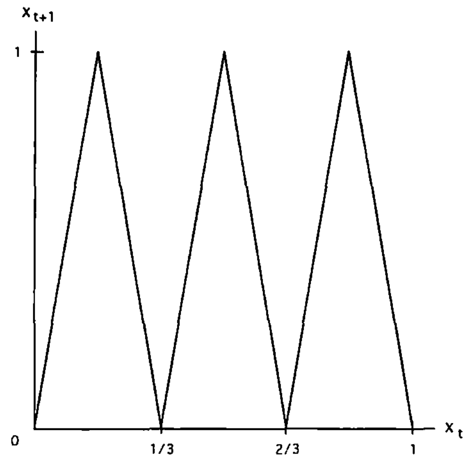

Fig. 2. The six-branch complete tent map.

An intertwining relation like (2.1) but with the p-adic map still holds, and following the same steps below that equation, we find that the even-order eigenpolynomials of U_{T_p} are proportional to $B_{2n}(x/2)$. In contrast to the incomplete tent maps, U_{T_p} does not commute with R. In fact, we have

$$U_{T_p} R = U_{T_p}$$

which shows that $U_{T_p} P^- = 0$. Thus, any odd function (with respect to the midpoint of the unit interval) is an eigenstate of U_{T_p} with eigenvalue 0. To form a polynomial basis, we choose here the odd-order Euler polynomials. Thus the even-order eigenpolynomials of any complete tent map are $C_{2n}(x) \equiv B_{2n}(x/2)$ and the odd-order eigenpolynomials are taken as $C_{2n-1}(x) \equiv E_{2n-1}(x)$.

The dual eigenstates $\tilde{C}_{2n}(x)$ of the even-order eigenpolynomials may be obtained in a straightforward way by considering the simplest case of the complete tent map with two branches. These states must be left eigenstates of U_T so that

$$\tilde{C}_{2n}(x)\; U_T = 2^{-2n} \tilde{C}_{2n}(x) \tag{3.1}$$

Also, they should be orthonormal with $C_{2m}(x) \equiv B_{2m}(x/2)$ as

$$\int_0^1 dx\; \tilde{C}_{2n}(x)\; B_{2m}\left(\frac{x}{2}\right) = \delta_{nm} \tag{3.2}$$

Acting with (3.1) on $B_{2m}(x)$ and then using that $U_T B_{2m}(x) = B_{2m}(x/2)$ gives

$$\int_0^1 dx\, \tilde{C}_{2n}(x)\, B_{2m}\left(\frac{x}{2}\right) = \int_0^1 dx\, 2^{-2n} \tilde{C}_{2n}(x)\, B_{2m}(x)$$

The left-hand side here is given from (3.2) and so $2^{-2n}\tilde{C}_{2n}(x)$ is the dual of $B_{2m}(x)$, i.e.,

$$\tilde{C}_{2n}(x) \equiv 2^{2n} \tilde{B}_{2n}(x)$$

As pointed out in the previous section, these states are orthogonal to all of the odd-order Euler polynomials. The states $\tilde{E}_{2n-1}(x)$, while forming an orthonormal set with the odd-order Euler polynomials, are not orthogonal to the even-order eigenstates $C_{2n}(x)$. To determine the correct duals here of the odd-order right eigenstates we utilize the completeness of the spectral decomposition of U_T so that

$$\tilde{C}_{2n-1}(x) \equiv \tilde{E}_{2n-1}(x) - \sum_{m=0}^{\infty} \left[\int_0^1 dx'\, \tilde{E}_{2n-1}(x')\, C_{2m}(x') \right] \tilde{C}_{2m}(x)$$

The map corresponding to the Frobenius–Perron operator RU_{Tp} starts with a branch of negative slope. The eigenvalues are the same as for U_{Tp} and the eigenstates are just R times the eigenstates given above.

4. CONCLUDING REMARKS

We have shown that the spectral decomposition of the Frobenius–Perron operator of tent maps may be easily determined by intertwining it with the r-adic map, for which the spectral decomposition is known, and utilizing symmetry considerations. Some of the results given here for tent maps hold for the general class of maps of the unit interval with piecewise-linear branches, each branch mapping onto the unit interval and all branches having the same absolute value of the slope. The P^+ projections of the even-order eigenpolynomials are always proportional to $B_{2n}(x)$. Also, the even-order left eigenstates themselves are proportional to $\tilde{B}_{2n}(x)$. These results for the general class may be obtained from intertwining relations with P^+.

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